EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS
*Earthquake Engng. Struct. Dyn.* **28**, 1483–1499 (1999)

# MAXIMUM SEISMIC DISPLACEMENT OF INELASTIC SYSTEMS BASED ON ENERGY CONCEPT

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## SUMMARY

The energy balance and energy input of Single-Degree-of-Freedom (SDF) systems under earthquake motion is studied for elastic and inelastic systems. The maximum displacement of an inelastic system is related to that of an elastic system having the same initial stiffness and mass by considering the earthquake energy input per cycle of oscillation.

With an assumption that the cyclic energy input is equal for both elastic and inelastic system for intermediate- and long-period systems, a simplified relation is suggested. Newmark's equal energy rule is shown to be the upper bound of the simplified relation; the lower bound of maximum response displacement is also derived in this paper. The numerical analysis results were mostly shown to fall between the proposed upper and lower bounds.

A separate approximate relation is proposed for short-period systems. The reason for divergence from the suggested relations is discussed for short-period systems. Copyright © 1999 John Wiley & Sons Ltd.

KEY WORDS: earthquake response; inelastic displacement; elastic displacement; SDF system; energy input; hysteresis energy dissipation

## 1. INTRODUCTION

Current seismic design generally allows the structure to undergo inelastic deformation during a strong earthquake motion. Therefore, the design lateral earthquake force can be significantly reduced from the maximum elastic input force. The amount of the design seismic force reduction depends on the deformation and energy dissipation capabilities of the structure. For this reason, it is important to estimate maximum response displacement during an earthquake and its relation with the energy dissipation of the system.

Based on the calculated maximum response characteristics of elastic and elasto-plastic Single-Degree-Freedom (SDF) systems having the same initial periods (same mass and stiffness),

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CCC 0098–8847/99/121483–17$17·50
Received 26 February 1998
Revised 5 January 1999
Accepted 4 May 1999

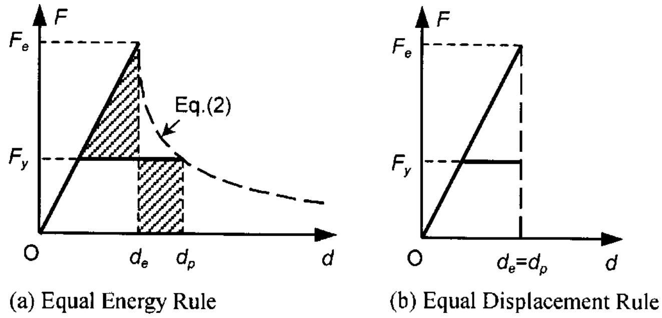



Figure 1. Newmark's design criteria

Veletsos and Newmark[1] proposed a set of well-known design criteria to determine the level of yield strength which could limit the maximum response to an acceptable ductility range, i.e.

(a) for a short-period system ($T < 0{\cdot}5$ sec), the yield resistance of an inelastic system must be greater than the value specified by equation (1),

$$F_y = \frac{F_e}{\sqrt{2\mu - 1}} \tag{1}$$

where $F_y$ is the yield resistance of the elasto-plastic system, $F_e$ the maximum elastic force response, $\mu(= d_a/d_y)$ the acceptable ductility factor for the elasto-plastic system, $d_y$ the yield displacement and $d_a$ the maximum acceptable displacement of the inelastic system.

The above approximate relation was derived on the basis of observation that maximum potential energy stored in the elastic and elasto-plastic systems was comparable at the maximum response displacement (equal shaded areas in Figure 1(a)), which was originally derived for the response of SDF systems under impulse loading. This criterion is called the equal energy rule.

Considering the limit state that the maximum response displacement $d_p$ of an inelastic system is equal to its acceptable displacement $d_a$, $d_p$ can be related to maximum elastic displacement $d_e$, i.e.

$$d_p = \frac{\mu}{\sqrt{2\mu - 1}} d_e \tag{2}$$

(b) for a long-period system ($T > 0{\cdot}5$ sec), the yield resistance must be greater than $F_y$ given in equation (3),

$$F_y = \frac{F_e}{\mu} \tag{3}$$

The above approximate relation was obtained on the basis of observation that maximum response displacements were comparable for elastic and elasto-plastic systems (Figure 1(b)). This criterion is called the equal displacement rule, i.e.

$$d_p = d_e \tag{4}$$

Although the Newmark's two design criteria are widely adopted in the seismic design codes of various countries, the two criteria were based on numerical experiments rather than on theory. Besides, the two design criteria are not always conservative.[2]

## 2. ENERGY BALANCE DURING EARTHQUAKE

The equation of motion of an SDF system is expressed as:

$$m(\ddot{x} + \ddot{x}_0) + c\dot{x} + f(x) = 0 \tag{5}$$

where, $m$ is the mass, $c$ the constant damping coefficient, $f(x)$ the restoring force of system, $x$ the displacement of mass relative to the base, and $\ddot{x}_0$ the acceleration of earthquake ground motion.

The infinitesimal work $\mathrm{d}W$ done by inertia force $m(\ddot{x} + \ddot{x}_0)$, damping force $c\dot{x}$ and restoring force $f(x)$ through absolute velocity $(\dot{x} + \dot{x}_0)$ over infinitesimal time interval $\mathrm{d}t$ can be expressed in the form of the following equation:

$$\mathrm{d}W = \{m(\ddot{x} + \ddot{x}_0) + c\dot{x} + f(x)\}(\dot{x} + \dot{x}_0)\,\mathrm{d}t = 0 \tag{6}$$

or, from the equation of motion,

$$\{m(\ddot{x} + \ddot{x}_0) + c\dot{x} + f(x)\}\dot{x}\,\mathrm{d}t = 0 \tag{7}$$

and

$$\{m(\ddot{x} + \ddot{x}_0) + c\dot{x} + f(x)\}\dot{x}_0\,\mathrm{d}t = 0 \tag{8}$$

The infinitesimal work $\mathrm{d}W$ must vanish at any instance; i.e. the energy must balance over any infinitesimal time interval for the equation of motion to hold. Normally, equation (7) is used when the earthquake energy input to a system is considered.

The energy balance is equivalent to the equilibrium of dynamic forces. The energy balance is considered for a finite time interval whereas the equilibrium is considered at each instance. As the response is influenced by the past history, the concept of energy balance may give some insight into the response characteristics.

The incremental work $\Delta W$ done over duration $t_1$ to $t_2$ is evaluated for equation (7),

$$\Delta W = \int_{t_1}^{t_2} \{(m\ddot{x} + \ddot{x}_0) + c\dot{x} + f(x)\}\dot{x}\,\mathrm{d}t = 0 \tag{9}$$

Rewriting the above equation, familiar energy expression is obtained

$$\int_{t_1}^{t_2} m\ddot{x}\dot{x}\,\mathrm{d}t + \int_{t_1}^{t_2} c\dot{x}^2\,\mathrm{d}t + \int_{t_1}^{t_2} f(x)\dot{x}\,\mathrm{d}t = -\int_{t_1}^{t_2} m\ddot{x}_0\dot{x}\,\mathrm{d}t \tag{10}$$

$$\left[\frac{1}{2}m\dot{x}^2\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} c\dot{x}^2\,\mathrm{d}t + \int_{t_1}^{t_2} f(x)\dot{x}\,\mathrm{d}t = -\int_{t_1}^{t_2} m\ddot{x}_0\dot{x}\,\mathrm{d}t \tag{11}$$

The first term on the left-hand side of equation (11) is the change in kinetic energy $\Delta E_{\mathrm{k}}$ over the time interval; the second term is incremental energy dissipation by the viscous damper $\Delta E_{\mathrm{d}}$; the third term is incremental strain energy absorption and hysteresis energy dissipation of the system $\Delta E_{\mathrm{f}}$; and the right-hand side of the equation is called an 'incremental earthquake energy input'

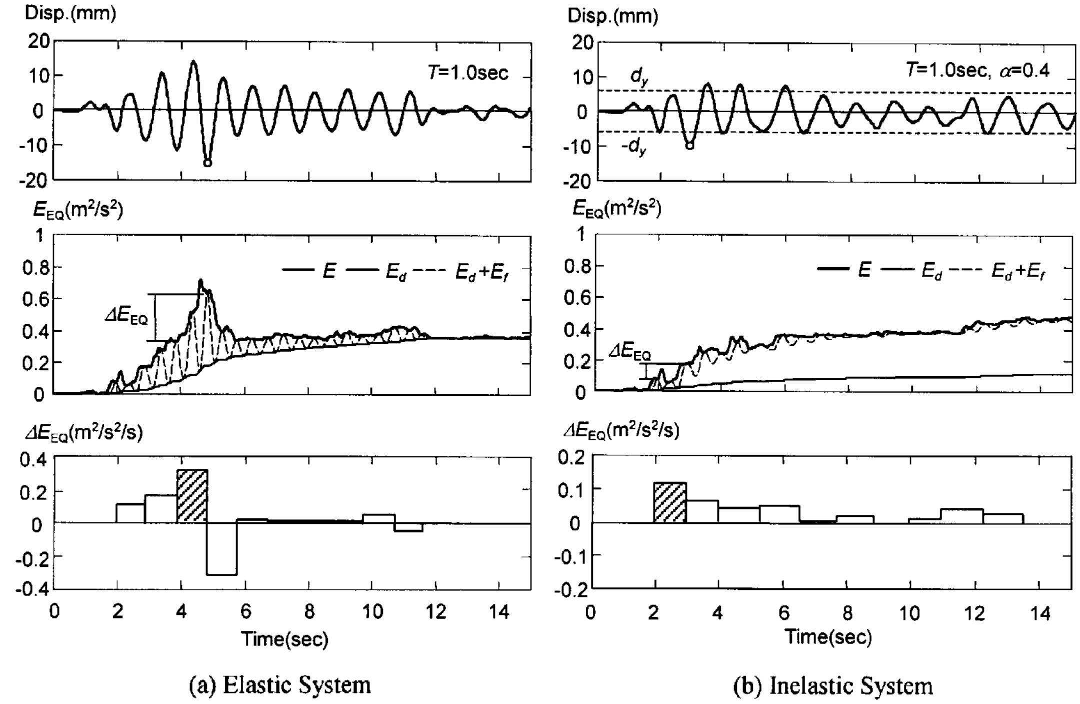



Figure 2. Displacement response, cumulative energy and incremental energy input per cyclic oscillation

$\Delta E_{\mathrm{EQ}}$, thus, the above equation can be written as

$$\Delta E_{\mathrm{k}} + \Delta E_{\mathrm{d}} + \Delta E_{\mathrm{f}} = \Delta E_{\mathrm{EQ}} \tag{12}$$

Evaluating equation (12) over the entire duration of an earthquake motion, the cumulative energy balance is obtained,

$$E_{\mathrm{k}} + E_{\mathrm{d}} + E_{\mathrm{f}} = E_{\mathrm{EQ}} \tag{13}$$

It should be noted that the cumulative earthquake energy input $E_{\mathrm{EQ}}$ is not a unique quantity of an earthquake motion, but is influenced by the response of a system.

Housner[3] and Akiyama[4] studied the energy balance over an entire duration of earthquake excitation and noted the stable nature of cumulative earthquake energy input in the response of elastic as well as inelastic SDF systems. Design criteria were proposed by equating the total energy input of an earthquake motion and the energy dissipation capacity of a structure.

Although the total energy balance reveals a cumulative hysteresis energy dissipation requirement or cumulative plastic deformation demand of an inelastic system, recent research indicates that the maximum response displacement depends on maximum incremental energy input.[5–7]

Figure 2 shows displacement response time histories, cumulative earthquake energy input $E_{\mathrm{EQ}}$ from the beginning of the earthquake motion and incremental earthquake energy input $\Delta E_{\mathrm{EQ}}$ during one cycle of oscillation for an elastic and an inelastic SDF system under the 1940 El Centro (NS) earthquake motion. The weight is taken to be 1000 kgf. A damping coefficient of the

inelastic system was assumed to vary proportional to instantaneous stiffness with the damping factor $h = 0{\cdot}02$. The resistance-deformation relation (hysteresis model) of the inelastic system was Clough hysteresis model[8] with the following modifications; i.e. (a) the unloading stiffness $k_r$ from the skeleton curve was reduced as the function of maximum ductility factor ($d_{max}/d_y$) from initial elastic stiffness $k$ as given in equation (14) with stiffness reduction index $\gamma = 0{\cdot}4$ and (b) the response point during reloading was modified to move towards an immediately preceding unloading point.[9] This modified Clough hysteresis model is suitable for reinforced concrete structure. The yield resistance was taken to be 0·4 times the maximum elastic response force. The initial natural period of both systems was $T = 1{\cdot}0$ sec. It should be pointed out here that the incremental earthquake energy input $\Delta E_{EQ}$ was calculated for peak-to-peak one-cycle displacement response. This incremental earthquake energy input $\Delta E_{EQ}$ is called 'cyclic earthquake energy input'.

$$k_r = k \left| \frac{d_{max}}{d_y} \right|^{-\gamma} \tag{14}$$

It should be noted that the cumulative earthquake input energy $E_{EQ}$ was completely dissipated by the viscous damper at the end of the earthquake motion in the elastic system (Figure 2(a)), while the cumulative input energy was dominantly dissipated by hysteresis in the inelastic system (Figure 2(b)). Although the maximum displacement occurs at different instances for the elastic and inelastic systems, the cyclic earthquake energy input $\Delta E_{EQ}$ was the largest (shade area in Figure 2) in the cycle immediately before the maximum displacement (cycle mark in displacement time history) for both systems. It is more reasonable to consider the cyclic energy input $\Delta E_{EQ}$ in estimating maximum response displacement.

## 3. DERIVATION OF SIMPLE ENERGY RELATION AT MAXIMUM RESPONSE

The concept of energy balance and energy input is used to study the relation between maximum displacements of an elastic and an inelastic system. Mass and initial stiffness are assumed to be same in a pair of the systems, resulting in the same initial natural period $T_n$.

The hysteresis model for the inelastic system is the Clough model as modified above. The skeleton relation is assumed elasto-plastic with yield deformation $d_y$ and yield resistance $F_y$. Owing to the stiffness reduction after yielding, an effective period $T_e$ of oscillation of the inelastic system becomes longer than the natural period $T_n$ of the corresponding elastic system.

Figures 3 and 4 show force-displacement relations and displacement response signals of the elastic and inelastic systems before the attainment of maximum displacement.

The elastic system follows displacement path $O \rightarrow A \rightarrow O \rightarrow B \rightarrow O \rightarrow A \rightarrow C$ before the maximum displacement is reached at point C (Figure 3). The inelastic system follows displacement path $O \rightarrow A' \rightarrow B \rightarrow C \rightarrow D \rightarrow A' \rightarrow E$ before the maximum displacement is attained at point E (Figure 4).

The following simplifying assumptions were made in the study; i.e.

(1) Both systems remain elastic before time $t_0$ at point O following the identical response history to the earthquake excitation; therefore, kinetic energy at time $t_0$ must be same for the elastic and inelastic systems.

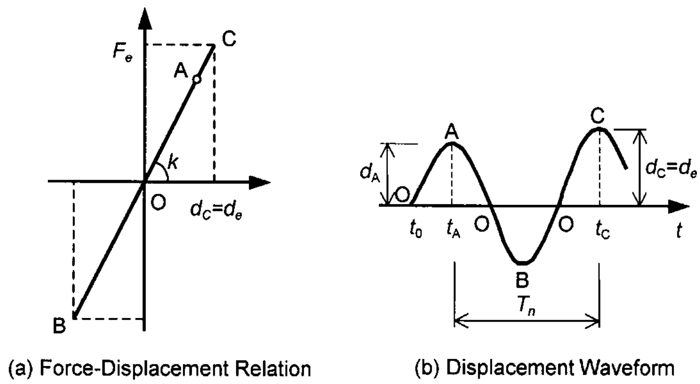



Figure 3. Elastic system

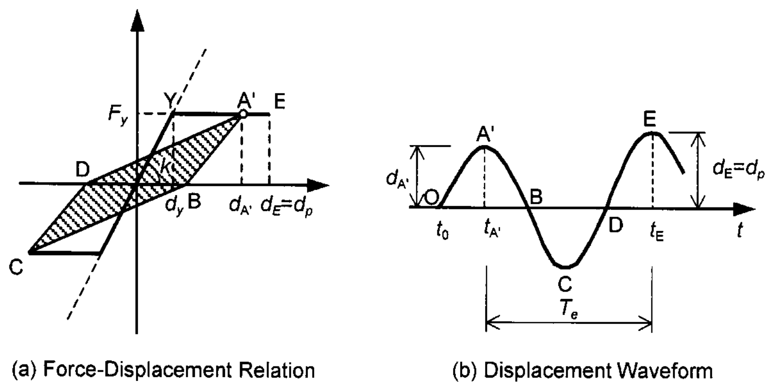



Figure 4. Inelastic system

(2) Incremental earthquake energy input from time $t_0$ to $t_A$ (or $t_{A'}$) at displacement peaks A (or A′) is comparable for both elastic and inelastic systems although the duration is slightly longer for the inelastic system.
(3) The energy dissipated by viscous damper is neglected.

The first assumption implies that the maximum response occurs in the first yielding cycle. The assumption may be reasonable in the response of an inelastic system with sufficient yield resistance under a near-field earthquake motion (e.g. the Kobe Marine Observatory record of the 1995 Hyogo-ken Nanbu Earthquake), but is not valid in the response to a far-field earthquake motion (e.g. the Hachinohe Harbor record in the 1968 Tokachi-oki Earthquake), especially for short-period systems and low-resistance systems, because such systems often exhibit large inelastic response before their maximum response. However, these assumptions were made to derive simple energy relations between elastic and inelastic SDF systems. The validity of the assumptions is examined by comparing the maximum response obtained by time-history analysis and the approximate method developed in this paper.

Based on assumption (2), the cumulative input energy $E_{EQ}(t_A)$ of the elastic system at time $t_A$ is equal to that of the inelastic system $E_{EQ}(t_{A'})$ at time $t_{A'}$. Noting that the velocity must be zero at displacement peak A (or A′) and neglecting energy dissipation by a viscous damper, equation (13), evaluated from $t = 0$ to $t_A$ (or $t = t_{A'}$), yields

$$E_f(t_A) = E_{EQ}(t_A) = E_{EQ}(t_{A'}) = E_f(t_{A'}) \tag{15}$$

Therefore, the strain energy $E_f(t_A) = \frac{1}{2}kd_A^2$ stored in the elastic system at time $t_A$ must be equal to strain energy $E_f(t_{A'}) = \frac{1}{2}kd_y^2 + (d_{A'} - d_y)F_y$ stored in the inelastic system at time $t_{A'}$.

Using equation (12) for the elastic system between time $t_A$ and $t_C$ ( $= t_A + T_n$) or between two successive displacement peaks $d_A$ and $d_C$ ( = maximum elastic displacement $d_e$), and noting that the velocity at the displacement peaks is null and neglecting energy dissipation by a viscous damper, the incremental strain energy $\Delta E_f(T_n)$ for the elastic system is,

$$\begin{aligned} \Delta E_f(T_n) &= E_f(t_C) - E_f(t_A) \\ &= \tfrac{1}{2}kd_C^2 - \tfrac{1}{2}kd_A^2 = \Delta E_{EQ}(T_n) \end{aligned} \tag{16}$$

where $\Delta E_{EQ}(T_n)$ is incremental earthquake energy input over one cycle of oscillation of the elastic system. Maximum displacement $d_e$ of an elastic system is attained at point C in Figure 3, strain energy $E_f(t_A)$ can be expressed as

$$\begin{aligned} E_f(t_A) &= E_f(t_C) - \Delta E_{EQ}(T_n) \\ &= \tfrac{1}{2}kd_e^2 - \Delta E_{EQ}(T_n) \end{aligned} \tag{17}$$

Similarly, for the inelastic system between time $t_{A'}$ and $t_E$ $(= t_{A'} + T_e)$ or between two successive peak displacements $d_{A'}$ and $d_E$ (= maximum inelastic displacement $d_p$), the incremental strain energy $\Delta E_f(T_e)$ is,

$$\begin{aligned} \Delta E_f(T_e) &= E_f(t_E) - E_f(t_{A'}) \\ &= \Delta W_{pl} + F_y(d_E - d_{A'}) = \Delta E_{EQ}(T_e) \end{aligned} \tag{18}$$

where $\Delta W_{pl}$ is the hysteresis energy dissipation during one cycle, equal to the shaded area in Figure 4(a). The dissipated hysteresis energy can be expressed as

$$\Delta W_{pl} = 4\beta F_y(d_p - d_y) \tag{19}$$

where $\beta$ is the hysteresis energy dissipation factor, equal to the ratio of shaded area in Figure 4(a) to $4F_y(d_p - d_y)$. $d_p$ is the maximum inelastic displacement response, equal to the displacement $d_E$ at time $t_E$. Equation (18) can be solved for $E_f(t_E)$ as

$$\begin{aligned} E_f(t_E) &= E_f(t_{A'}) + 4\beta F_y(d_p - d_y) + F_y(d_p - d_{A'}) \\ &= \tfrac{1}{2}kd_y^2 + F_y(d_{A'} - d_y) + F_y(d_p - d_{A'}) + 4\beta F_y(d_p - d_y) \\ &= \tfrac{1}{2}kd_y^2 + F_y(d_p - d_y) + 4\beta F_y(d_p - d_y) \end{aligned} \tag{20}$$

and also,

$$\begin{aligned} E_f(t_{A'}) &= E_f(t_E) - \Delta E_{EQ}(T_e) \\ &= \tfrac{1}{2}kd_y^2 + F_y(d_p - d_y) + 4\beta F_y(d_p - d_y) - \Delta E_{EQ}(T_e) \end{aligned} \tag{21}$$

With equation (15), equations (17) and (21) can be combined to yield the following relation:

$$\tfrac{1}{2}kd_y^2 + F_y(d_p - d_y) + 4\beta F_y(d_p - d_y) - \Delta E_{EQ}(T_e) = \tfrac{1}{2}kd_e^2 - \Delta E_{EQ}(T_n) \tag{22}$$

By introducing a strength reduction factor $\alpha$ defined as the ratio of yield resistance $F_y$ to maximum elastic resistance $F_e$ at point C in Figure 3(a), and noting that $F_e = kd_e$, $F_y = kd_y$, and $d_y = \alpha d_e$, the following relations are derived:

$$d_p = \frac{\lambda + (8\beta + 1)\alpha^2}{2(4\beta + 1)\alpha} d_e \tag{23}$$

$$\lambda = 1 + \left[\frac{\Delta E_{EQ}(T_e)}{\Delta E_{EQ}(T_n)} - 1\right]\frac{\Delta E_{EQ}(T_n)}{(1/2)kd_e^2} \tag{24}$$

Input energy factor $\lambda$ depends on maximum cyclic energy inputs $\Delta E_{EQ}(T_n)$ and $\Delta E_{EQ}(T_e)$ during the cycle of oscillation of elastic and inelastic systems, respectively, before maximum displacement is attained. Maximum displacement $d_p$ of the inelastic system is related to maximum elastic displacement $d_e$, yield resistance reduction factor $\alpha$, hysteresis energy dissipation factor $\beta$, and input energy factor $\lambda$.

## 4. CHARACTERISTICS OF MAXIMUM CYCLIC INPUT ENERGY $\Delta E_{EQ}$

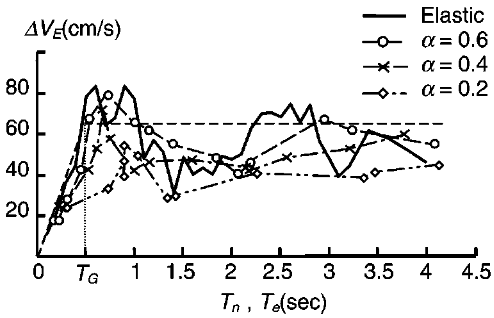
Non-linear earthquake response is calculated for a series of elastic and inelastic SDF systems having same initial periods under the 1940 El Centro (NS) earthquake motion. Maximum cyclic input energy $\Delta E_{EQ}(T_n)$ and $\Delta E_{EQ}(T_e)$ before the maximum displacement for elastic and inelastic systems are evaluated and expressed in the form of equivalent input velocity $\Delta V_E$ using the following relation:

$$\Delta V_E = \sqrt{\frac{2\Delta E_{EQ}}{m}} \tag{25}$$

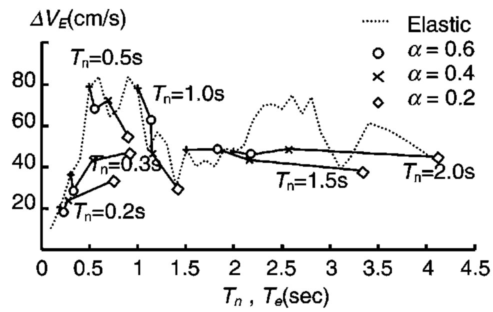
In Figure 5(a), the equivalent input velocity spectrum is plotted for natural period $T_n$ of elastic systems or equivalent periods $T_e$ of inelastic systems. The solid line represents the equivalent input velocity spectrum for elastic systems, whereas others represent equivalent input velocities calculated for inelastic systems having yield resistance $F_y = (0{\cdot}6, 0{\cdot}4, 0{\cdot}2)F_e$. The equivalent input velocity spectrum of an elastic system is generally larger than that of inelastic systems. Thus, the equivalent input velocity spectrum of elastic systems can be approximately regarded as the upper bound to that of inelastic systems.

Figure 5(b) shows the variation of equivalent input velocities $\Delta V_E$ of elastic and inelastic systems having the same initial period. The yield resistance of an inelastic system was reduced by factor $\alpha$ from the maximum elastic response force. The equivalent period $T_e$ of inelastic systems increases with decreasing $\alpha$ due to increasing plastic deformation. For a range of periods larger than 0·5 sec, the $\Delta V_E - T_e$ relation of inelastic systems generally agrees with the elastic equivalent input velocity spectrum; i.e. cyclic energy input $\Delta E_{EQ}$ can be approximately evaluated from the elastic equivalent velocity spectrum at period $T_n$ for an elastic system and period $T_e$ for an inelastic system.




(a) Equivalent Velocity $\Delta V_E$ Spectra




(b) Influence of Yield Resistance Reduction Factor on Maximum Equivalent Velocity $\Delta V_E$

Figure 5. Equivalent input velocity $\Delta V_{\mathrm{E}}$

For a period range shorter than 0·5 sec in Figure 5(b), the equivalent input velocity of an elastic system having natural period of $T_{\mathrm{n}} = T_{\mathrm{e}}$ is much larger than that of an inelastic system, especially when the value of $\alpha$ is small. But it is conservative for design purpose to use the equivalent input velocity of an elastic system as the upper bound to that of the inelastic system.

Equivalent input velocity spectrum is shown to resemble the corresponding velocity response spectrum of linearly elastic systems or equivalent velocity spectra of cumulative energy input.[4,6] The equivalent input velocity spectrum of elastic systems may be approximately idealized by bi-linear relations represented by short-dashed lines intersected at a corner period $T_{\mathrm{G}}$ in Figure 5(a), and the following characteristics may be pointed out; i.e.

(a) In a short-period range ($T_{\mathrm{n}} < T_{\mathrm{G}}$), equivalent input velocity increases linearly with periods $T_{\mathrm{n}}$ or $T_{\mathrm{e}}$.
(b) In a long-period range ($T_{\mathrm{n}} > T_{\mathrm{G}}$), equivalent input velocity is stable.

The corner period $T_{\mathrm{G}}$ varies from one earthquake motion to another, influenced by the earthquake source mechanism, path of earthquake motion and surface geology at the site. However, the corner period can be roughly estimated to be around 0·4–0·6 s for most earthquake motions.

## 5. CHARACTERISTICS OF HYSTERESIS ENERGY DISSIPATION FACTOR $\beta$

The hysteresis energy dissipation factor $\beta$ is defined as the ratio of one hysteresis loop area ABCDA to the area AFCGA ($= 4F_{\mathrm{y}}(d_{\mathrm{p}} - d_{\mathrm{y}})$) of an inelastic system as shown in Figure 6. The factor varies from 0 (no hysteresis energy dissipation with $\gamma = 1{\cdot}0$ in equation (14)) to 1·0 (fully elasto-plastic with $\gamma = 0{\cdot}0$) in general cases.

When the modified Clough model is used to represent the hysteric characteristics of a reinforced concrete structure, the unloading stiffness reduction index $\gamma$ in equation (14) is normally taken to be around $0{\cdot}4 \sim 0{\cdot}5$.[2] Therefore, the factor $\beta$ for reinforced concrete structures is calculated to be 0·27–0·37 for a ductility factor $\mu = d_{\mathrm{p}}/d_{\mathrm{y}}$ ranging from 1·5 to 4·0 (Figure 7). The factor $\beta$ increases with ductility factor $\mu = d_{\mathrm{p}}/d_{\mathrm{y}}$, but the value is relatively stable. For a range of ductility factors from 1·5 to 4·0, factor $\beta$ may be assumed to be 0·33.

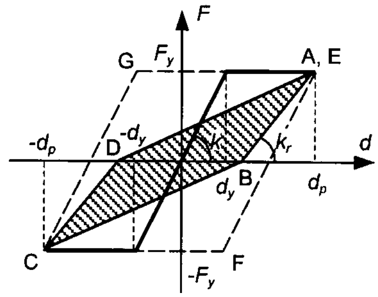



Figure 6. Simplified hysteresis cycle

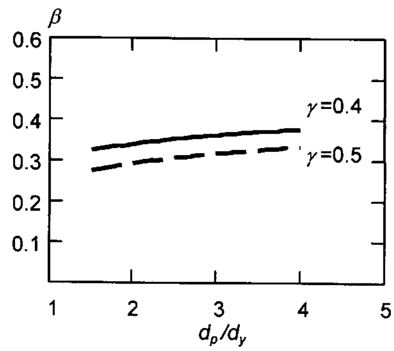



Figure 7. Hysteresis energy dissipation factor $\beta$

## 6. SIMPLIFIED RELATION FOR MAXIMUM RESPONSE DISPLACEMENT

Simplified relations of maximum response displacement of an elastic and an inelastic system are derived on the basis of the following two assumptions about the cyclic energy input and equivalent velocity spectrum; i.e.

(1) Cyclic energy input $\Delta E_{EQ}$ can be approximately estimated from an elastic equivalent velocity spectrum at period $T_n$ for an elastic system, and period $T_e$ for an inelastic system.

(2) Equivalent input velocity spectrum of elastic system can be idealized by a bilinear relation with a corner period $T_G$.

The two assumptions are based on the observation made in Figure 5. The validity of these assumptions is examined by comparing the maximum response obtained by time-history analysis and the approximate method developed in this paper.

### 6.1. *For system period $T_n > T_G$*

The equivalent input velocity is constant for a period range $T_n > T_G$ and maximum cyclic energy inputs are the same for an elastic and an inelastic system; i.e.

$$\Delta E_{EQ}(T_n) = \Delta E_{EQ}(T_e) \tag{26}$$

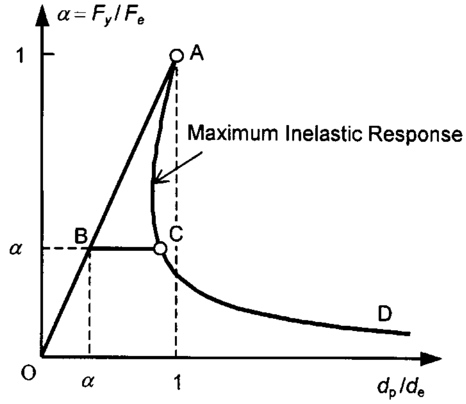



Figure 8. Maximum response of elastic and inelastic systems

hence, the input energy factor $\lambda$ becomes unity. Equation (23) can be simplified as

$$d_p = \frac{1{\cdot}0 + (8\beta + 1)\alpha^2}{2(4\beta + 1)\alpha} d_e \tag{27}$$

It can be observed in equation (27) that for the same value of yield resistance reduction factor $\alpha$, the maximum inelastic displacement $d_p$ decreases with an increasing hysteresis energy dissipation factor $\beta$. Consequently, the upper bound of equation (27) is obtained if the hysteresis energy dissipation factor $\beta$ is zero. For $\beta = 0$, equation (27) is reduced to equation (28), which is the same as Newmark's equal energy criterion if the yield resistance reduction factor is defined by equation (1).

$$d_p = \frac{1 + \alpha^2}{2\alpha} d_e \tag{28}$$

The lower bound of equation (27) is obtained for $\beta = 1{\cdot}0$, at which the hysteresis energy dissipation capacity reaches maximum; i.e.

$$d_p = \frac{1 + 9\alpha^2}{10\alpha} d_e \tag{29}$$

Figure 8 compares maximum response of elastic and inelastic systems. Point A represents the maximum response of the elastic system with maximum elastic response force $F_e$ and displacement $d_e$. The vertical and horizontal axes are normalized with respect to $F_e$ and $d_e$, respectively. Point C represents the maximum response of the inelastic system having yield resistance $F_y$ and maximum inelastic response displacement $d_p$. Line OA represents the elastic force–displacement relation, while bilinear line OBC represents the skeleton of inelastic force–displacement relation. The yield strength and displacement at point B are $F_y = \alpha F_e$, and $d_y = \alpha d_e$. For a series of inelastic systems with different yield resistance reduction factors $\alpha$, curve ACD is obtained to represent the maximum inelastic response.

Figure 9 compares the upper bound (equation (28)), the lower bound (equation (29)) and equation (27) with $\beta = 0{\cdot}33$ by varying the yield resistance reduction factor $\alpha$. The vertical line at $d_p/d_e = 1{\cdot}0$ in Figure 9 represents the Newmark's equal displacement rule. It can be known from

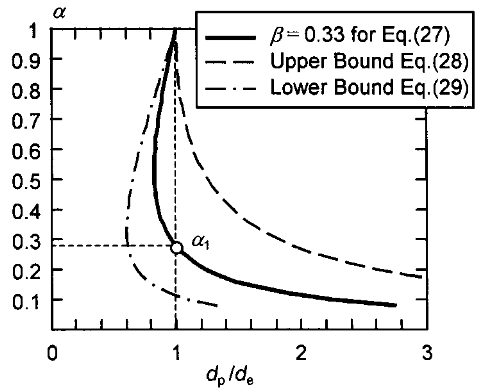



Figure 9. Comparison of equation (27) with $\beta = 0{\cdot}33$, equations (28) and (29)

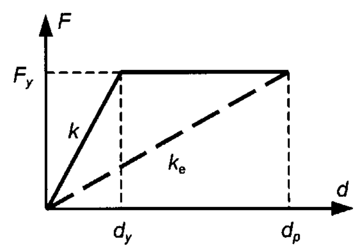



Figure 10. Initial stiffness and equivalent stiffness of yielding cycle

Figure 9 that if an inelastic system is provided with a yield resistance reduction factor higher than a limiting value $\alpha_1$ for a given factor $\beta$, the maximum inelastic displacement is smaller than the corresponding elastic displacement attributable to hysteresis energy dissipation, so that the Newmark's equal displacement rule holds approximately. For a factor $\alpha$ less than the limiting value $\alpha_1$, the Newmark's equal displacement rule does not hold even in an intermediate- to long-period range.

### 6.2. *For system period $T_n < T_G$*

For a range of period $T_n < T_G$, the equivalent input velocity $\Delta V_E$ increases linearly with period, hence, maximum cyclic energy input $\Delta E_{EQ}$ $(= \frac{1}{2} m\Delta V_E^2)$ is proportional to the square of period. Noting $T_n = 2\pi\sqrt{m/k}$,

$$\frac{\Delta E_{EQ}(T_e)}{\Delta E_{EQ}(T_n)} = \frac{\Delta V_E(T_e)^2}{\Delta V_E(T_n)^2} = \frac{T_e^2}{T_n^2} = \frac{k}{k_e} \tag{30}$$

From Figure 10, $k$ and $k_e$ can be replaced by $k = F_y/d_y$ and $k_e = F_y/d_p$, and noting $d_y = \alpha d_e$, equation (30) can be written as

$$\frac{\Delta E_{EQ}(T_e)}{\Delta E_{EQ}(T_n)} = \frac{k}{k_e} = \frac{d_p}{d_y} = \frac{d_p}{\alpha \cdot d_e} \tag{31}$$

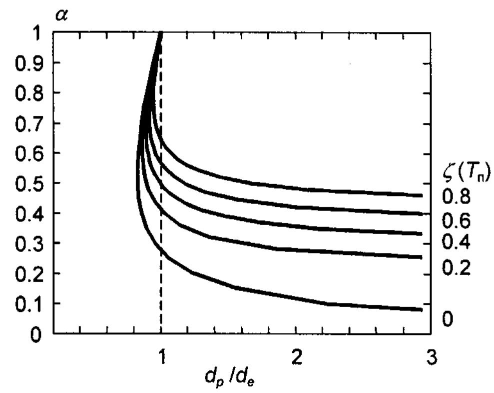



Figure 11. Variation of equation (33) with $\zeta(T_n)$

Substituting equation (31) into equations (24) and (23), the following relation is obtained:

$$d_p = \frac{1 + [d_p/\alpha \cdot d_e - 1](\Delta E_{EQ}(T_n)/(1/2)kd_e^2) + (8\beta + 1)\alpha^2}{2(4\beta + 1)\alpha} d_e \tag{32}$$

Therefore, the maximum inelastic displacement $d_p$ can be expressed as follows:

$$d_p = \frac{1 - \zeta(T_n) + (8\beta + 1)\alpha^2}{2(4\beta + 1)\alpha - \zeta(T_n)/\alpha} d_e \tag{33}$$

$$\zeta(T_n) = \frac{\Delta E_{EQ}(T_n)}{(1/2)k d_e^2} = \left[\frac{T_n}{2\pi}\frac{\Delta V_E(T_n)}{d_e}\right]^2 \tag{34}$$

Equation (33) is plotted for $\beta = 0{\cdot}33$ in Figure 11 by varying $\zeta(T_n)$. When $\zeta(T_n) = 0{\cdot}0$, equations (33) and (27) become identical. It should be noted that the relation between the maximum displacement ratio $d_p/d_e$ of an elastic and an inelastic system varies significantly with $\zeta(T_n)$, especially when $\zeta(T_n)$ is small ($\zeta(T_n)$ less than 0·4). However, the relation becomes stable for a larger value of $\zeta(T_n)$ ($\zeta(T_n)$ more than 0·4). For a given value of $\zeta(T_n)$, the maximum inelastic response can be significantly larger than the elastic response at small values of $\alpha$, especially for larger value of $\zeta(T_n)$.

## 7. RELIABILITY OF PROPOSED SIMPLE RELATIONS

To examine the reliability of simple relations on the basis of a series of the simplifying assumptions, non-linear earthquake response of a series of SDF systems was calculated. In the analysis, the modified Clough model was used with an unloading stiffness reduction index $\gamma = 0{\cdot}4$ (equation (14)). The initial natural period $T_n$ was varied from 0·15 to 3·5 sec. The 1940 El Centro (NS) and the 1952 Taft (EW) earthquake records were used as input excitation.

For an initial period $T_n$, the linearly elastic response was first calculated to determine the maximum elastic response force $F_e$ and displacement $d_e$. The response of a series of inelastic systems with the same initial period was calculated by varying factor $\alpha$ ( $= F_y/F_e$) from 0·1 to 1·0.

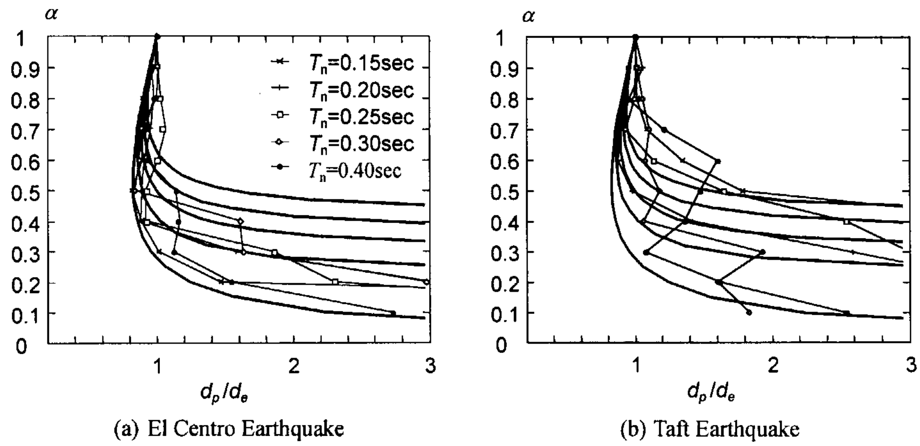



Figure 12. Comparison for systems with period $T_n = 0{\cdot}15$–$0{\cdot}4$ sec. Note: the solid lines without symbol represent the results of equation (33) in both figures, the value of $\zeta(T_n)$ is 0, 0·2, 0·4, 0·6 and 0·8 from bottom line to top line

Table I. Calculated value of $\zeta(T_n)$

| initial period In (sec) | El Centro (NS) | Taft (EW) |
|---|---|---|
| | $\zeta(T_n)$ | $\zeta(T_n)$ |
| $T_n = 0{\cdot}15$ | 0·34 | 0·81 |
| $T_n = 0{\cdot}20$ | 0·50 | 0·48 |
| $T_n = 0{\cdot}25$ | 0·72 | 0·77 |
| $T_n = 0{\cdot}30$ | 0·79 | 0·48 |
| $T_n = 0{\cdot}40$ | 0·64 | 0·76 |

Viscous damping coefficient was assumed to vary proportional to instantaneous stiffness with initial damping factor of 0·02.

The reliability of the proposed simple expressions, equations (27) and (33), is examined by comparing them with the calculated maximum response ratio $d_p/d_e$.

### 7.1. For an initial natural period $T_n = 0{\cdot}15$–$0{\cdot}4$ sec

In Figure 12, the calculated $\alpha - d_p/d_e$ curves of inelastic systems are compared with the approximate relations of equation (33) with $\beta = 0{\cdot}33$ and varying values of $\zeta(T_n)$, such as 0, 0·2, 0·4, 0·6 and 0·8. The value $\zeta(T_n)$ was evaluated by equation (34) assuming a bilinear equivalent input velocity spectrum $\Delta V_E$ for each earthquake motion (Table I); the value of $\zeta(T_n)$ varies significantly with natural periods for an earthquake motion and also from an earthquake motion to another even at the same natural period. The value of $\zeta(T_n)$ generally lies between 0·4 and 0·8 for El Centro (NS) and Taft (EW) motions.

The calculated inelastic response agrees reasonably with the approximate relation (equation (33)) for a range of $\alpha > 0{\cdot}6$. However, the approximate relation is poor for a small value of $\alpha$ as

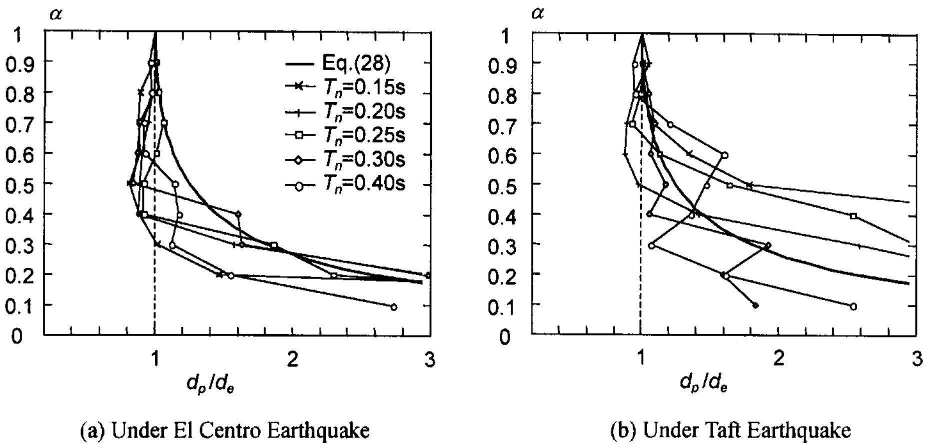



Figure 13. Equation (28) and maximum response of short-period systems

expected by the simplifying assumption because large inelastic response occurs prior to the maximum response in a short-period and low-resistance system. Further study is necessary to estimate the maximum inelastic response on the basis of the energy balance in a short-period range. The upper bound to the maximum inelastic response may be obtained from Figure 12 by using $\zeta(T_n) = 0{\cdot}60$ for El Centro motion and $\zeta(T_n) = 1{\cdot}0$ for Taft motion. If the maximum response of a short-period and low-resistance system is very sensitive to the characteristics of a ground motion, then it is necessary to use different values of $\zeta(T_n)$ for a ground motion.

The reliability of the Newmark's equal energy rule is examined in a short-period range by comparing the calculated maximum inelastic response with the relation of equation (28) in Figure 13. For El Centro (NS) motion, the Newmark's equal energy rule gives the upper bound to the maximum inelastic response, but the rule does not give the upper bound to the response under Taft (EW) motion. This may be attributable to the fact that the energy input sometimes can be significantly different for an inelastic and an elastic system in a short-period range.

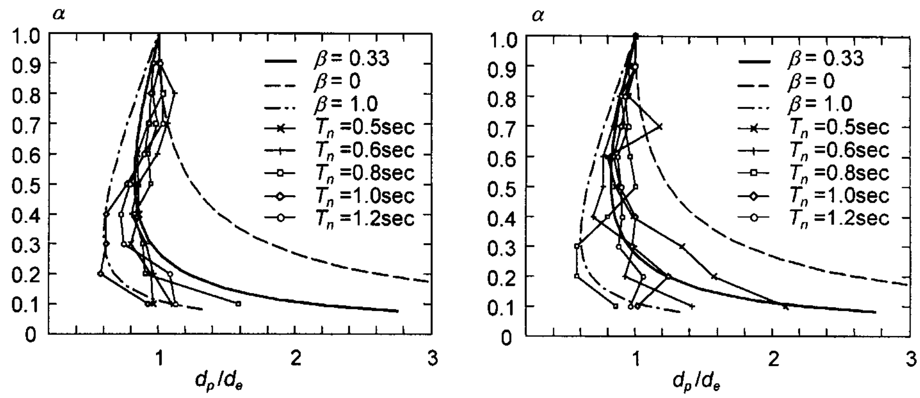
### 7.2. *For an initial natural period $T_n = 0{\cdot}5$–$1{\cdot}2$ sec*

Under the El Centro (NS) earthquake motion (Figure 14(a)), the calculated $\alpha - d_p/d_e$ curves lie between equation (27) with $\beta = 0{\cdot}33$ and the upper bound (equation (28)) for $\alpha > 0{\cdot}5$, and lie between equation (27) with $\beta = 0{\cdot}33$ and the lower bound (equation (29)) for $\alpha < 0{\cdot}5$. The calculated $\alpha - d_p/d_e$ curves agree reasonably well with equation (27) with $\beta = 0{\cdot}33$ under the Taft (EW) earthquake motion (Figure 14(b)). It can be noticed that $d_p/d_e$ is greater than 1·0 for systems with smaller yield resistance reduction factor $\alpha$.

### 7.3. *For an initial natural period of $T_n = 1{\cdot}5$–$3{\cdot}5$ sec*

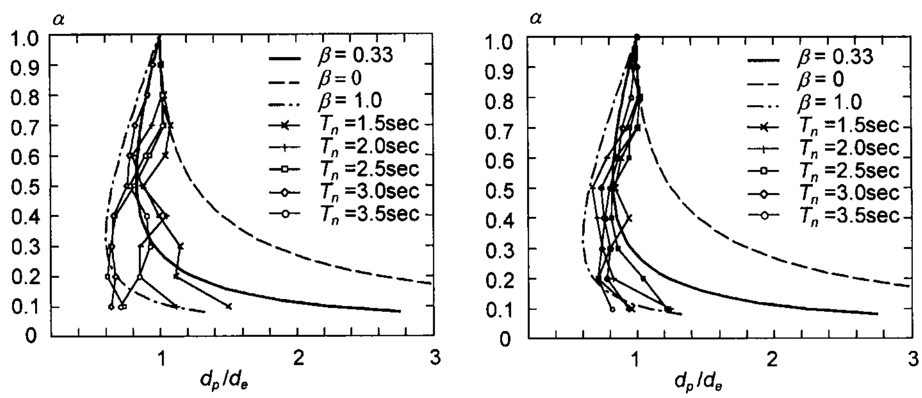
The calculated $\alpha - d_p/d_e$ curves (Figure 15) fall between equation (27) with $\beta = 0{\cdot}33$ and the upper bound (equation (28)) for $\alpha > 0{\cdot}5$, and between equation (27) with $\beta = 0{\cdot}33$ and the lower bound (equation (29)) for $\alpha < 0{\cdot}5$ under both earthquake motions. There are also cases that $d_p/d_e$ is greater than 1·0 for systems with smaller yield resistance reduction factor $\alpha$.




(a) Under El Centro Earthquake (b) Under Taft Earthquake

Figure 14. Comparison for systems with period $T_n = 0{\cdot}5$–$1{\cdot}2$ sec




(a) Under El Centro Earthquake (b) Under Taft Earthquake

Figure 15. Comparison for systems with period $T_n = 1{\cdot}5$–$3{\cdot}5$ sec

The discrepancy between the calculated response and equation (27) using $\beta = 0{\cdot}33$ in Figures 14 and 15 can be explained below:

(1) the assumption that the maximum response occurs in the first yielding cycle does not hold for an inelastic system with low resistance,

(2) a low-resistance system develops an inelastic response increasing factor $\beta$ much larger than 0·33,

(3) a large-resistance system develops an inelastic response decreasing factor $\beta$ less than 0·33 and

(4) cyclic energy input increases with the increase in effective period especially for a large inelastic deformation.

## 9. CONCLUSIONS

Maximum response displacement of inelastic systems under earthquake motion is studied on the basis of the energy concept. Simple energy relations are derived to relate the maximum response of an elastic and inelastic system in a short- and a long-period range on the basis of two assumptions: (a) the maximum response occurs in the first yielding cycle, (b) the equivalent input velocity spectrum, representing maximum cyclic input energy, is idealized by a bilinear relation with a corner period $T_G$. The reliability of the assumption was examined by comparing the calculated maximum response with the approximate response.

In a long-period range ($T_n > T_G$), the simple relation equation (27) with $\beta = 0{\cdot}33$ gives a reasonable estimate of the maximum inelastic response on the basis of elastic response. The Newmark's equal displacement rule is found to be reasonable for systems with higher yield resistance. The upper bound of equation (27) is shown to be the Newmark's equal energy rule.

In a short-period range ($T_n < T_G$), the proposed simple relation explains the trend of inelastic maximum response with yield resistance, but does not give reasonable estimates especially for low-resistance systems. The poor agreement is attributed to poor reliability of the assumptions for short-period and low-resistance systems. Moreover, the Newmark's equal energy rule does not give the upper bound to the inelastic response in this period range.

### ACKNOWLEDGEMENTS

The authors appreciate the financial assistance of Ichikawa Fund that enabled the first author to stay at the University of Tokyo for the research related this paper. The authors also acknowledge the constructive opinions by the reviewers.

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